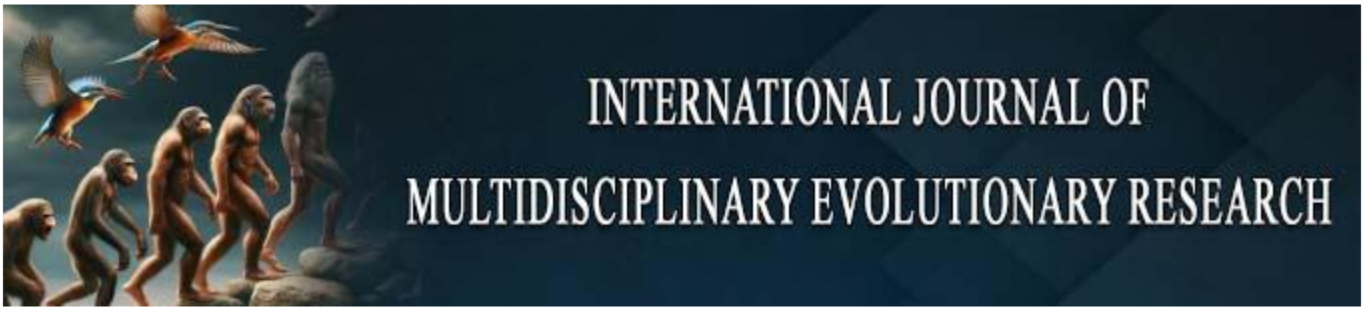

Graph-Theoretic Methods for Studying Topological Networks and Simplicial Complexes

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Abstract
Graph theory and topology provide complementary mathematical frameworks for studying complex systems. Graphs describe systems through vertices and edges, while topology investigates global structural properties such as connectivity, holes, cycles, continuity, and higher-dimensional relationships. In many modern applications, however, pairwise relationships are not sufficient to represent the full structure of a system. Simplicial complexes extend graphs by incorporating higher-order interactions among three or more elements, thereby enabling a richer topological description. This paper examines graph-theoretic methods for studying topological networks and simplicial complexes. It focuses on graph connectivity, paths, cycles, components, cliques, adjacency structures, incidence relations, clique complexes, nerves, boundary operators, Betti numbers, and higher-order connectivity. The study adopts a theoretical and analytical methodology and shows how graphs serve as the one-dimensional foundation of simplicial complexes, while simplicial complexes generalize graph-based representations to higher dimensions. The paper also discusses applications in social networks, communication systems, biological networks, sensor networks, image analysis, and topological data analysis. Particular attention is given to the transition from graphs to simplicial complexes through clique expansion and filtration-based constructions. The analysis indicates that graph-theoretic techniques are useful for identifying local and global structural properties, whereas simplicial methods are necessary when higher-order relationships and topological holes must be represented. The paper concludes that the integration of graph theory and simplicial topology provides a powerful mathematical framework for studying complex networked systems.

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1. Introduction

Graph theory is one of the most widely used mathematical tools for representing relational structures. A graph is usually written as

$$G = (V, E),$$

where V denotes the set of vertices and E denotes the set of edges. Vertices may represent individuals, devices, locations, biological entities, data points, or other objects, while edges indicate relationships between pairs of vertices.

Graph models have been successfully applied to communication systems, transportation, social networks, biological systems, computer networks, and image structures. Their main strength lies in their simplicity. A complex system can often be reduced to a set of objects and pairwise relationships among them.

However, many real systems contain interactions that involve more than two entities simultaneously. For example, a group of three individuals may participate in the same collaborative activity, several neurons may participate in a functional interaction, or multiple sensors may collectively cover a common region. Such interactions cannot be represented completely by ordinary pairwise edges.
This limitation leads naturally to simplicial complexes.
A simplicial complex is constructed from simplices. A 0-simplex is a vertex, a 1-simplex is an edge, a 2-simplex is a filled triangle, a 3-simplex is a tetrahedron, and higher-dimensional simplices extend this idea further.
Thus, graph theory can be viewed as the one-dimensional foundation of simplicial topology.

The connection can be represented as

Vertices → Edges → Triangles → Tetrahedra → Higher-dimensional simplices.

Topological networks combine ideas from network science and topology. They seek to understand not only which elements are connected but also how these connections organize themselves into global structures, cycles, cavities, communities, and higher-order interactions.
The purpose of the present paper is to examine how graph-theoretic methods can be used for studying topological networks and simplicial complexes and to explain why higher-dimensional structures are important in modern network analysis.

2. Objectives of the Study
The major objectives of the study are:
1. To explain the relationship between graph theory and simplicial topology.
2. To examine graph-theoretic methods for analyzing connectivity in topological networks.
3. To explain how graphs can be transformed into simplicial complexes.
4. To study cliques, cycles, incidence relations, and higher-order connectivity.
5. To identify major advantages and limitations of these approaches.

3. Research Methodology
The present research adopts a theoretical, descriptive, and analytical methodology.
The study is based on established mathematical concepts from graph theory, algebraic topology, combinatorial topology, network science, and topological data analysis. Definitions and mathematical constructions are analyzed to show how graph structures can be extended into simplicial complexes.

Particular attention is given to
- connected graphs,
- paths and cycles,
- graph components,
- cliques,
- clique complexes,
- simplicial complexes,
- boundary operators,
- homology groups,
- Betti numbers, and
- filtration-based network structures.

The study does not use fabricated experimental data. Instead, it presents conceptual examples and mathematical formulations to explain the relationship among graph theory, topological networks, and simplicial complexes.

4. Fundamental Concepts of Graph Theory
A graph

$$G = (V, E)$$

consists of a vertex set

$$V = \{v_1, v_2, \dots, v_n\}$$

and an edge set

$$E \subseteq \left\{\{v_i, v_j\}, : v_i v_j \in V\right\}.$$

Two vertices are adjacent if they share an edge.

4.1 Degree
The degree of a vertex v, denoted by

$$\deg(v),$$

is the number of edges incident to it.
The degree indicates the local level of connectivity of a vertex.

4.2 Path
A path is a sequence

$$v_0, v_1, \dots, v_k$$

such that

$$(v_i, v_{i+1}) \in E$$

for each consecutive pair.

4.3 Connected Graph
A graph is connected if a path exists between every pair of vertices.
If a graph is disconnected, it can be divided into connected components.

The number of connected components may be written as

$$c(G).$$

For a connected graph,

$$c(G) = 1.$$

4.4 Cycle
A cycle is a closed path in which the starting and ending vertex are the same.
Cycles are especially important in topological network

analysis because they may indicate loop-like structures.

5. Topological Networks

A topological network may be understood as a network analyzed not only through conventional graph measures but also through structural properties inspired by topology.

Traditional graph analysis focuses on quantities such as

- degree,
- shortest path,
- centrality,
- clustering,
- connected components, and
- network density.

A topological analysis additionally considers

- loops,
- holes,
- cavities,
- higher-order connectivity,
- structural persistence, and
- dimensional features.

Thus, a graph-based network can serve as the starting point for a topological representation.

6. Simplicial Complexes

A simplex is the basic building block of a simplicial complex. A k-simplex contains $k+1$vertices.

Therefore:

$$\begin{aligned}&0\text{-simplex} = \text{vertex},\\ &1\text{-simplex} = \text{edge},\\ &2\text{-simplex} = \text{filled triangle},\end{aligned}$$

and

$$3\text{-simplex} = \text{tetrahedron}.$$

A simplicial complex $K$is a collection of simplices satisfying two fundamental conditions.
First, every face of a simplex in Kalso belongs to K.
Second, the intersection of any two simplices is either empty or a common face.

For example, if

$$[v_1, v_2, v_3]$$

is a 2-simplex, then its edges

$$[v_1, v_2],\ [v_2, v_3],\ [v_1, v_3]$$

and its vertices must also belong to the complex.

7. Graphs as One-Dimensional Simplicial Complexes

Every simple graph can be viewed as a one-dimensional simplicial complex.
Vertices are interpreted as 0-simplices and edges as 1-simplices.

Thus,

$$G = (V, E)$$

can be associated with

$$K_G = V \cup E.$$

However, a graph alone does not distinguish between an empty triangular cycle and a filled triangular region.
Suppose three vertices satisfy

$$\begin{aligned}&(v_1, v_2) \in E,\\ &(v_2, v_3) \in E,\end{aligned}$$

And

$$(v_1, v_3) \in E.$$

The graph contains a triangle.
In a simplicial complex, this may either remain a one-dimensional cycle or be filled with the 2-simplex

$$[v_1, v_2, v_3].$$

This distinction is fundamental because a triangular loop and a filled triangle have different topological properties.

8. Cliques and Clique Complexes

A **clique** in a graph is a subset of vertices in which every pair is connected by an edge.
A k-clique contains $k$mutually adjacent vertices.
Given a graph G, its **clique complex**, also called its flag complex, is formed by treating each clique as a simplex.
Thus:

- every vertex produces a 0-simplex,
- every edge produces a 1-simplex,
- every 3-clique produces a 2-simplex,
- every 4-clique produces a 3-simplex.

If

$$C = \{v_1, v_2, v_3\}$$

is a 3-clique, then the clique complex contains

$$[v_1, v_2, v_3].$$

The clique-complex construction provides one of the most important bridges between graph theory and simplicial topology.

9. Connectivity in Simplicial Complexes

Connectivity in a simplicial complex can often be examined initially through its 1-skeleton.
The **1-skeleton** of a simplicial complex consists of all its vertices and edges.
If the 1-skeleton is connected, then the simplicial complex is connected.
This means standard graph algorithms such as breadth-first search and depth-first search can be applied to the 1-skeleton to determine basic connectivity.
Thus,

Simplicial Complex → 1-Skeleton → Graph Connectivity.

This is an important computational advantage because complicated higher-dimensional complexes can still be analyzed using established graph algorithms for their basic connectedness.

10. Higher-Order Connectivity

Ordinary graphs represent pairwise interactions.
Suppose three vertices A, B, and $C$are pairwise connected.
Graphically:

$$A - B,$$
$$B - C,$$
$$A - C.$$

A graph indicates three pairwise relationships.

A 2-simplex

$$[A, B, C]$$

can instead represent a genuine three-way relationship involving all three simultaneously.
This distinction is important in applications.
For example, in a social network, three people who communicate pairwise may not necessarily participate in the same group interaction. A simplicial representation can distinguish pairwise connections from a true three-person interaction.
Thus simplicial complexes provide a more expressive model of higher-order networks.

11. Boundary Operators

Boundary operators are fundamental to algebraic-topological analysis.

For a k-simplex

$$[v_0, v_1, \ldots, v_k],$$

its boundary is formally expressed as
where $\widehat{v_i}$means that $v_i$is omitted.

For a 1-simplex,

$$\partial_1[v_0, v_1] = [v_1] - [v_0].$$

For a 2-simplex,

$$\partial_2[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1].$$

A fundamental identity is

$$\partial_k \partial_{k+1} = 0.$$

In words, the boundary of a boundary is zero.
This property forms the algebraic foundation for homology.

12. Homology and Topological Features

Homology provides a method for identifying holes of different dimensions.
The k-th homology group is defined as

$$H_k = \frac{\ker \partial_k}{\operatorname{im} \partial_{k+1}}.$$

The dimension of the homology group is the k-th Betti number:

$$\beta_k = \operatorname{rank}(H_k).$$

Important interpretations include:
β_0 = number of connected components,
β_1 = number of independent one-dimensional holes,

and
β_2 = number of two-dimensional cavities or voids.

These quantities extend conventional graph analysis.
Graph theory is particularly effective for $\beta_0$and graph cycles, but simplicial homology provides a more rigorous distinction between cycles that bound filled regions and cycles that represent actual holes.

13. Euler Characteristic

Another important invariant is the Euler characteristic.
For a finite simplicial complex,
where $f_k$represents the number of k-simplices.
For a two-dimensional complex,

$$\chi = f_0 - f_1 + f_2,$$

where:

- f_0= number of vertices,
- f_1= number of edges,
- f_2= number of triangles.

The Euler characteristic is also related to Betti numbers:

$$\chi = \beta_0 - \beta_1 + \beta_2 - \cdots .$$

This relationship provides an important connection between combinatorial counting and topological structure.

14. Graph Cycles and Topological Holes

For a finite graph, the cycle-space dimension is

$$|E| - |V| + c,$$

where $c$is the number of connected components.
For a connected graph,

$$\beta_1 = |E| - |V| + 1$$

when the graph is viewed strictly as a one-dimensional simplicial complex.
However, if some graph cycles are filled by 2-simplices in a higher-dimensional complex, they no longer represent nontrivial one-dimensional homology classes.
Consider a triangle.
As a graph alone, it has one cycle.
Therefore,

$$\beta_1 = 1.$$

If the triangular region is filled with a 2-simplex, its boundary is filled and the hole disappears.

Then,

$$\beta_1 = 0.$$

This example clearly demonstrates why simplicial complexes provide richer information than graphs alone.

15. Nerve Complexes

Another important construction is the nerve of a family of sets.
Let

$$\mathcal{U} = \{U_1, U_2, \dots, U_n\}.$$

Each set U_i becomes a vertex.
An edge

$$[v_i, v_j]$$

is included whenever

$$U_i \cap U_j \neq \emptyset.$$

A 2-simplex is included whenever

$$U_i \cap U_j \cap U_k \neq \emptyset.$$

Higher-dimensional simplices correspond to higher-order nonempty intersections.
Thus, the nerve records the entire intersection pattern of the family.
Its 1-skeleton is simply the intersection graph.
Hence,

$$\text{Intersection Graph} \subset \text{Nerve Complex}.$$

This provides another important relationship between graph theory and topology.

16. Vietoris-Rips Complexes

One of the most widely used simplicial constructions in topological data analysis is the Vietoris-Rips complex.
Suppose a set of points is given with a distance function d.
For a scale parameter ϵ, connect two vertices whenever

$$d(v_i, v_j) \leq \epsilon.$$

This produces a graph.
Every clique in this graph is then filled with a simplex.
The resulting complex is denoted by

$$VR(X, \epsilon).$$

Thus,

$$\text{Point Cloud} \to \text{Distance Graph} \to \text{Clique Complex} \to \text{Vietoris-Rips Complex}.$$

As ϵ increases, more edges and simplices appear.
This produces a filtration.

17. Filtrations and Persistent Homology

A filtration is a nested sequence

$$K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_m.$$

At different filtration levels, components may merge, loops may appear, and cavities may form or disappear.
Persistent homology tracks these features across scales.
A topological feature that exists only for a very short range may represent noise, while a long-lasting feature may indicate meaningful structure.
Persistent homology therefore extends graph-theoretic network analysis by incorporating scale.

The general framework is

$$\text{Weighted Network} \to \text{Filtration} \to \text{Simplicial Complexes} \to \text{Persistent Homology} \to \text{Topological Features}.$$

18. Application in Social Networks

Social networks provide an important example of higher-order interactions.
Traditional graph models represent people as vertices and relationships as edges.
However, social interaction often occurs in groups.
A triangle among three people may represent three pairwise friendships, while a 2-simplex may represent a joint collaborative group.
Higher-dimensional simplices can represent larger teams, committees, research groups, or communication groups.
This approach enables researchers to distinguish pairwise networks from genuinely collective interactions.

19. Application in Communication Networks

Communication systems are commonly analyzed using graph connectivity.
Routers, computers, or communication nodes become vertices, while communication links become edges.
Graph methods can determine:

- shortest paths,
- connectivity,
- articulation points,
- bridges,
- redundancy, and
- network resilience.

A simplicial representation can extend this analysis by representing higher-order communication among groups of nodes.
For example, simultaneous multi-node interactions in wireless systems or distributed computing may be more naturally modeled using simplices than ordinary edges.

20. Application in Sensor Networks

Sensor networks provide a particularly natural connection between topology and simplicial complexes.
Suppose each sensor covers a region.
Vertices correspond to sensors.
An edge can indicate overlap between the coverage regions of two sensors.
A 2-simplex can indicate common overlap among three sensor regions.
The resulting nerve complex can provide information about coverage and potential gaps.
Topological holes may indicate uncovered regions.
Therefore,

$$\text{Sensor Coverage} \rightarrow \text{Intersection Relations} \rightarrow \text{Simplicial Complex} \rightarrow \text{Coverage Analysis.}$$

This is a practical example where higher-order topology provides information that simple graph connectivity alone may not capture.

21. Application in Biological Networks

Biological networks include neural networks, protein-interaction networks, metabolic systems, gene-regulatory structures, and brain-connectivity networks.
Traditional graph analysis measures connectivity among biological entities.
Simplicial methods provide a way to examine higher-order interactions.
For instance, groups of brain regions may demonstrate coordinated activity that cannot be adequately represented as independent pairwise relationships.
Simplicial complexes therefore offer a mathematical framework for investigating collective biological structures.

22. Application in Image Analysis

Images can also be represented as networks.
Pixels or regions become vertices, and adjacency relations become edges.
Graph methods can identify:

- connected image regions,
- boundaries,
- skeletons,
- components, and
- neighborhood relationships.

When higher-dimensional cells or complexes are added, topological features such as holes can be studied more rigorously.
Cubical complexes are often especially natural for pixel and voxel data, while simplicial complexes can be obtained through suitable triangulation or point-cloud constructions.
Thus, graph theory provides the first discrete representation, while topology extends the structural analysis.

23. Application in Topological Data Analysis

Topological Data Analysis (TDA) is one of the most important modern application areas.
Complex datasets can first be represented through similarity graphs.
The graphs are subsequently transformed into simplicial complexes.
Persistent homology then identifies stable structural properties.
This framework is useful because conventional statistics may describe averages, variances, and correlations but may not explicitly reveal loops, holes, or multi-scale connectivity structures.
TDA therefore complements traditional methods by emphasizing the shape of data.

24. Graph-Theoretic Algorithms for Simplicial Analysis

Graph algorithms remain useful even after a simplicial complex has been constructed.
Breadth-first search and depth-first search determine connected components of the 1-skeleton.
Clique-finding algorithms can identify candidate higher-dimensional simplices.
Shortest-path algorithms can investigate network distances.
Spanning trees provide minimal connectivity structures.
Cycle-detection methods can identify one-dimensional cyclic structures.
Matrix-based graph representations also support simplicial calculations.
An adjacency matrix is defined by

$$A_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

Simplicial complexes extend this representation through boundary and incidence matrices.
Thus, matrix algebra provides a computational bridge between graph methods and algebraic topology.

25. Advantages of the Integrated Approach

Combining graph theory and simplicial topology provides several advantages.
First, graph theory offers simple and computationally efficient representations.
Second, simplicial complexes extend graph models to higher-order interactions.
Third, connectivity analysis can often be performed through the graph 1-skeleton.
Fourth, homology distinguishes trivial cycles from genuine topological holes.
Fifth, the framework applies naturally to weighted and dynamic networks.
Sixth, filtration methods make it possible to study structures at multiple scales.
Finally, the approach can integrate pure mathematical analysis with computational and data-driven applications.

26. Limitations and Challenges

Several limitations must also be recognized.
The first is computational complexity. A graph with many cliques can generate an extremely large simplicial complex.
For example, a clique involving $n$vertices contains many faces, causing rapid growth in the number of simplices.
Second, pairwise graph data may not accurately imply genuine higher-order interaction. Constructing a clique complex assumes that pairwise relations justify filling the corresponding simplex. In some applications, this assumption may be inappropriate.
Third, high-dimensional homology calculations can be computationally expensive.
Fourth, network noise can create artificial cycles and simplices.
Fifth, selecting filtration parameters may influence persistent-homology results.
Therefore, the mathematical construction should always reflect the scientific meaning of the underlying data.

27. Major Findings

The theoretical analysis leads to several important findings.
Graph theory forms the one-dimensional foundation of simplicial-complex analysis.

The 1-skeleton of a simplicial complex allows ordinary graph methods to determine basic connectivity.
Clique complexes provide a systematic method for converting pairwise graph structures into higher-dimensional simplicial structures.
Graph cycles and topological holes are not necessarily identical because a cycle may be filled by higher-dimensional simplices.
Homology and Betti numbers provide mathematically rigorous measures of connected components, loops, and cavities.
Nerve and Vietoris-Rips complexes demonstrate how graph-theoretic relationships can generate topological structures.
Persistent homology extends static network analysis by tracking topological features across different scales.
Finally, higher-order simplicial models may provide more realistic representations than ordinary graphs when multi-way relationships are intrinsic to a system.

28. Future Scope

Future research can extend graph-theoretic topology in several directions.
One important direction is **higher-order network science**, where simplicial complexes and hypergraphs are used to model interactions involving multiple entities.
Another direction concerns dynamic simplicial networks in which vertices, edges, and higher-dimensional simplices change over time.
The integration of persistent homology with machine learning and graph neural networks offers additional scope.
Other promising research areas include:

- brain network analysis,
- biological interaction systems,
- social group dynamics,
- wireless communication,
- sensor coverage,
- image topology,
- explainable artificial intelligence, and
- large-scale topological data analysis.

Efficient algorithms for constructing and simplifying large simplicial complexes will also be essential.

29. Conclusion

Graph theory and topology are closely related mathematical disciplines, particularly in the study of connectivity and structural organization. Graph theory represents systems using vertices and edges and provides efficient methods for analyzing paths, components, cycles, and network relationships. However, pairwise relationships alone may not be sufficient for systems involving multi-way interactions.
Simplicial complexes extend graph theory by introducing higher-dimensional simplices. A graph may therefore be interpreted as the one-dimensional skeleton of a richer topological structure. Clique complexes, nerve complexes, and Vietoris-Rips complexes provide important constructions through which graph data can be transformed into topological representations.
The study demonstrates that graph connectivity is particularly useful for determining connected components, while homology and Betti numbers extend the analysis to loops and cavities. The distinction between an unfilled graph cycle and a filled simplicial triangle illustrates why higher-dimensional information is essential.
Graph-theoretic and simplicial methods have important applications in social systems, communication networks, biological structures, sensor networks, image analysis, and topological data analysis. Persistent homology further enables these structures to be analyzed across multiple scales.
It can therefore be concluded that graph theory provides the computational and combinatorial foundation for the study of topological networks, while simplicial complexes provide the higher-dimensional framework required for representing collective interactions and topological structure. Their integration offers a powerful and increasingly important methodology for the mathematical analysis of complex systems.

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